

BLOW UP OF A CYCLE IN LOTKA-VOLTERRA TYPE EQUATIONS WITH COMPETITION-COOPERATION TERMS AND QUASI-LINEAR SYSTEMS

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We consider systems where a cycle born via the Hopf bifurcation blows up to infinity as a parameter ranges over a finite interval. Two examples demonstrating this effect are presented: planar Lotka-Volterra type systems with a competition-cooperation term and quasi-linear higher order equations.

Keywords: Hopf bifurcation; Global branch of cycles; Lotka-Volterra systems; Competition-cooperation term; Degree of mapping; Conley index.

1. Introduction

We consider one scenario of transformation of a cycle of a differential equation, which we call the blow up. In this scenario, a cycle born via the Hopf bifurcation grows to infinity as a parameter ranges over a finite segment. From another perspective, in the product of the phase space and the parameter axis there is a branch of cycles connecting the equilibrium and infinity. We first discuss the existence of such a branch for planar differential equations where the proof can be based on the Poincare theorem. A Lotka-Volterra type system with a competition-cooperation term is considered as an example. Then we discuss the existence of a branch of cycles stretching from zero to infinity for a class of higher order quasilinear equations: this theorem continues the results of Refs. 1,2. The results for planar systems are presented in the next section. Section 3 contains the main result for higher order equations. We briefly sketch some points of the proofs.

2. Planar systems

Consider a planar system

$$x' = f(x, y; \lambda), \quad y' = g(x, y; \lambda) \quad (1)$$

with a parameter $\lambda \in [0, 1]$, where the functions f, g are continuously differentiable with respect to the phase variables x, y and continuous with respect to the set of all arguments. We say that the system has a branch of cycles connecting the origin and infinity if for any bounded open domain $G \ni (0, 0)$ of the phase plane the system has a cycle (for some $\lambda \in [0, 1]$) that belongs to the closure \bar{G} of G and touches the boundary ∂G of G . This definition, close to the classical weak definition of continuous branches of fixed points³, is discussed in Ref. 1.

Proposition 2.1. *Suppose that the following conditions hold:*

- (i) *$f(0, 0; \lambda) = 0$ and the origin is the only equilibrium point of system (1) for all $\lambda \in [0, 1]$;*
- (ii) *the Jacobi matrix $J = J(\lambda)$ of system (1) at the origin is invertible for all $\lambda \in [0, 1]$;*
- (iii) *the spectrum of J is in the open right half-plane for $\lambda = 0$ and in the open left half-plane for $\lambda = 1$;*
- (iv) *the system does not have cycles for $\lambda = 0$ and $\lambda = 1$.*

Then system (1) has a branch of cycles connecting the origin and infinity.

Condition (ii) ensures that no equilibrium branches from the origin, which agrees with condition (i). According to condition (iii), the origin changes stability as λ ranges over the segment $[0, 1]$. Conditions (ii) and (iii) guarantee that the Andronov–Hopf bifurcation occurs within this segment. Condition (iv) is satisfied, for example, if the system has a global Lyapunov function for $\lambda = 0$ and $\lambda = 1$. Under the assumptions of Proposition 2.1 the system does not have homoclinic orbits; the cycles surround the origin.

As an example, consider the Lotka–Volterra type equations

$$x' = x(a - by), \quad y' = y(-c + dx + f(y; \lambda)); \quad x, y > 0 \quad (2)$$

with positive parameters a, b, c, d . The last term f in the predator equation accounts for competition ($f < 0$) or cooperation ($f > 0$) in the predator population. We assume that the behavior of the predator can depend on its number: it competes if the population number is above a certain threshold $\bar{y} = \bar{y}(\lambda)$ and starts to cooperate when the population falls below the threshold, hence $f(y; \lambda) < 0$ for $y > \bar{y}(\lambda)$ and $f(y; \lambda) > 0$ for $y < \bar{y}(\lambda)$. For example, $f = \arctan y - \lambda y$ with $\lambda \in (0, 1)$.

Suppose that $c > f(ab^{-1}; \lambda)$ for all λ (the condition $c > f$ means that the predator extincts in the absence of prey). Then system (2) has a unique positive equilibrium $(x_*, y_*) = (d^{-1}(c - f(ab^{-1}; \lambda)), ab^{-1})$ and Proposition

2.1 can be applied after the logarithmic coordinate transformation and the shift of the equilibrium to zero. The equilibrium (x_*, y_*) is stable if $f'_y < 0$ and unstable if $f'_y > 0$ at $y = y_*$, where $f'_y = \partial f / \partial y$ is the partial derivative of $f = f(y; \lambda)$. Hence conditions (i) – (iii) of Proposition 2.1 are satisfied if

$$f'_y(y_*; 0) > 0, \quad f'_y(y_*; 1) < 0. \quad (3)$$

If in addition

$$(y - y_*)(f(y; 0) - f(y_*; 0)) > 0, \quad y > 0, \quad y \neq y_*, \quad (4)$$

$$(y - y_*)(f(y; 1) - f(y_*; 1)) < 0, \quad y > 0, \quad y \neq y_*, \quad (5)$$

then condition (iv) is also satisfied, because in this case $V = (x - x_* \ln x)d + (y - y_* \ln y)b$ is a Lyapunov function of system (2) for $\lambda = 0, 1$ with $\dot{V} = (y - y_*)(f(y; \lambda) - f(y_*; \lambda))b$. Hence, relations (3) – (5) ensure the existence of a branch of cycles connecting the equilibrium and infinity for system (2) (the cycles lie in the positive quadrant $x, y > 0$ where the system is defined). In particular, these relations hold if f strictly increases for $\lambda = 0$ and strictly decreases for $\lambda = 1$, as in the above example $f = \arctan y - \lambda y$. Numerical simulations confirm that a stable positive cycle born via the Hopf bifurcation blows up to infinity for this f and demonstrate the same effect for other competition-cooperation terms, such as $f = y - \lambda y^2$ or $f = y^2 - \lambda y^3$, included in the equation for predator, or prey, or both.

To prove Proposition 2.1, one can first note that under its conditions the equilibrium can not have eigenvalues of different sign and consequently the system does not have homoclinic orbits. Using the Poincare theorem, one derives from this fact that if the system has an orbit γ in a bounded domain \bar{G} with $\gamma \cap \partial G \neq \emptyset$, then it also has a cycle $\mathcal{C} \subset \bar{G}$ with $\mathcal{C} \cap \partial G \neq \emptyset$. To complete the proof by contradiction, assume that there is no such a cycle and hence no such an orbit γ for some bounded domain $G \ni 0$. Consequently, $S_\lambda \cap \partial G = \emptyset$, where S_λ denotes the invariant set of the system in the domain \bar{G} (note that $0 \in S_\lambda$). Therefore \bar{G} is an isolating neighborhood for S_λ and the Conley index $\text{Ind } S_\lambda$ of S_λ with respect to \bar{G} is defined⁴. Because the system has no homoclinic orbits and, by condition (iv), there is no cycles for $\lambda = 0, 1$, the Poincare theorem implies that $S_0 = S_1 = 0$. Moreover, condition (iii) implies $\text{Ind } S_0 \neq \text{Ind } S_1$. This, however, contradicts the invariance of the Conley index under homotopic transformation of the vector field: $\text{Ind } S_\lambda$ should be the same for all λ for any isolating neighborhood \bar{G} of S_λ . Given any open bounded $G \ni 0$, this contradiction proves the existence of a cycle $\mathcal{C} \subset \bar{G}$ with $\mathcal{C} \cap \partial G \neq \emptyset$ for some λ , i.e. the conclusion of the proposition.

3. Quasi-linear higher order equations

Consider the equation

$$L\left(\frac{d}{dt}; \lambda\right)x = f(x; \lambda), \quad (6)$$

where $L(p; \lambda) = p^\ell + a_1(\lambda)p^{\ell-1} + \cdots + a_0(\lambda)$ is a polynomial with continuously differentiable coefficients. Assume that the continuous function $f(x; \lambda)$ satisfies $f(0; \lambda) \equiv 0$ and the global Lipschitz estimates

$$|f(x_1; \lambda) - f(x_2; \lambda)| \leq k|x_1 - x_2|, \quad |f(x; \lambda_1) - f(x; \lambda_2)| \leq l|x||\lambda_1 - \lambda_2|; \quad (7)$$

hence the equation has the zero solution $x \equiv 0$ for all λ . Define the matrix

$$J(w, \lambda) = \begin{pmatrix} \Re L'_\lambda(\lambda; wi) & -\Im L'_p(\lambda; wi) \\ \Im L'_\lambda(\lambda; wi) & \Re L'_p(\lambda; wi) \end{pmatrix}.$$

We say that equation (6) has a Lipschitz continuous branch of cycles connecting zero and infinity if there are Lipschitz continuous functions $\lambda(r)$, $w(r)$ with values in segments $[\lambda_-, \lambda_+]$, $[w_-, w_+]$ ($w_- > 0$) such that for every $r > 0$ equation (6) with $\lambda = \lambda(r)$ has a periodic solution $x_r(t) = x(t; r)$ of the period $2\pi/w(r)$, the function $x(t; r)$ is Lipschitz continuous in r and

$$\|x_r\|_C \rightarrow 0 \text{ as } r \rightarrow 0, \quad \|x_r\|_C \rightarrow \infty \text{ as } r \rightarrow \infty.$$

Theorem 3.1. *Assume that for some $q > 0$ the relation $|L(wi; \lambda)| \leq q$ defines a simply connected bounded domain D_q on the plane (w, λ) , the equation $L(wi; \lambda) = 0$ has a unique solution (w_0, λ_0) in D_q , the matrix $J(w, \lambda)$ is nondegenerate in D_q , and $L(nwi; \lambda) \neq 0$ in D_q for any integer $n \neq \pm 1$. Then there are sufficiently small $k, l > 0$ such that equation (6) with any function f satisfying the estimates (7) has a Lipschitz continuous branch of cycles connecting zero and infinity.*

The method of the proof of Theorem 3.1 leads to explicit estimates of the Lipschitz coefficients k, l , which ensure the existence of the branch of cycles connecting zero and infinity.

A natural parameter r is the amplitude of the first harmonics of the periodic solution. Theorem 3.1 can be proved by contraction mapping principle. To construct the corresponding mapping, let us first note that for any $(w, \lambda) \in D_q$ the differential operator $L(w \frac{d}{dt}; \lambda)$ with the 2π -periodic boundary conditions is invertible on the codimension 2 subspace \mathbb{E} of $\mathbb{L}^2 = \mathbb{L}^2(0, 2\pi)$ which is orthogonal to $\sin t$ and $\cos t$ (this operator, however, is not invertible on the whole space \mathbb{L}^2 , because $L(w_0 i; \lambda_0) = 0$). Secondly, the planar map $(w, \lambda) \mapsto (\Re L(wi; \lambda), \Im L(wi; \lambda))$ is invertible

on D_q . Now, denote by P_s and P_c the orthogonal projectors onto $\sin t$ and $\cos t$ in \mathbb{L}^2 , define the orthogonal projector $Q = I - P_s - P_c$ onto $\mathbb{E} = Q\mathbb{L}^2$ and consider the space of triples $(u, v, y) \in \mathbb{R} \times \mathbb{R} \times Q\mathbb{L}^2$ with the norm $\|(u, v, y)\| = \sqrt{u^2 + v^2 + \|y\|_{L^2}^2}$, where $y = y(t)$. In this space, for each value of the parameter $r > 0$ consider the mapping

$$(u, v, y) \mapsto A_r(u, v, y) = r^{-1}(P_s f(x(t); \lambda), P_c f(x(t); \lambda), Qf(x(t); \lambda)),$$

where $x = x(t)$ and λ are defined by the relations

$$u = \Re L(wi; \lambda), \quad v = \Im L(wi; \lambda), \quad x(t) = r(\pi^{-1/2} \sin t + h(t)), \quad (8)$$

$$L\left(w \frac{d}{dt}; \lambda\right)h(t) = y(t), \quad h(0) = h(2\pi), \quad h'(0) = h'(2\pi), \quad h \in Q\mathbb{L}^2. \quad (9)$$

Due to the invertibility of these relations mentioned above, the mapping A_r is well-defined for all $u^2 + v^2 \leq q^2$ and all $y \in Q\mathbb{L}^2$. The definition of A_r ensures that every fixed point of A_r defines a 2π -periodic solution $x = x(t)$ of the equation $L(w \frac{d}{dt}; \lambda)x = f(x; \lambda)$ by the formulas (8), (9) and hence a $2\pi/w$ -periodic solution $x(wt)$ of equation (6). The proof is completed by showing that if k, l are sufficiently small then A_r is a contraction on the ball $\|(u, v, y)\| \leq q$, which is invariant for A_r , for each $r > 0$.

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